REPORTS@SCM
AN ELECTRONIC JOURNAL OF THE
SOCIETAT CATALANA DE MATEMÀTIQUES

# A negative result for hearing the shape of a triangle: a computer-assisted proof

*Gerard Orriols Giménez
ETH Zürich.
gerard.orriols@math.ethz.ch
∗Corresponding author

**Resum** *(CAT)*
En aquest article demostrem que existeixen dos triangles diferents pels quals el primer, segon i quart valor propi del Laplacià amb condicions de Dirichlet coincideixen. Això resol una conjectura proposada per Antunes i Freitas i suggerida per la seva evidència numèrica. La prova és assistida per ordinador i utilitza una nova tècnica per tractar l'espectre d'un l'operador, que consisteix a combinar un Mètode d'Elements Finits per localitzar aproximadament els primers valors propis i controlar la seva posició a l'espectre, juntament amb el Mètode de Solucions Particulars per confinar aquests valors propis a un interval molt més precís.

**Abstract** *(ENG)*
We prove that there exist two distinct triangles for which the first, second and fourth eigenvalues of the Laplace operator with zero Dirichlet boundary conditions coincide. This solves a conjecture raised by Antunes and Freitas and suggested by their numerical evidence. We use a novel technique for a computer-assisted proof about the spectrum of an operator, which combines a Finite Element Method, to locate roughly the first eigenvalues keeping track of their position in the spectrum, and the Method of Particular Solutions, to get a much more precise bound of these eigenvalues.

**Keywords:** *computer-assisted proof, Laplace eigenvalues, spectral geometry, Finite Element Method, Method of Particular Solutions.*
**MSC (2010):** *35P05, 35R30, 58J53.*
**Received:** *September 2, 2020.*
**Accepted:** *October 30, 2020.*

*Acknowledgement*
The author would like to thank *Reports@SCM* for the opportunity to publish his Bachelor's thesis as a consequence of being awarded the Noether prize from the SCM.

Societat Catalana de Matemàtiques
Institut d'Estudis Catalans

# 1. Introduction

The main result of the thesis is the following theorem.

**Theorem 1.1.** *The first, second and fourth eigenvalues of the Laplace operator on an Euclidean triangle with null Dirichlet boundary conditions are not enough to determine it up to isometry.*

This is a conjecture proposed by Antunes and Freitas in [1], suggested by numerical evidence, but a rigorous proof was required. The Dirichlet eigenvalues of the Laplace operator for a triangle $\Omega$ are real numbers $\lambda$ such that there is a nonzero smooth function $u$ defined on $\Omega$ and continuous on $\overline{\Omega}$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is a well known fact that the set of such $\lambda$ forms an increasing sequence $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ whose only limit point is $\infty$, and that the corresponding eigenfunctions $u_j$ form an orthonormal basis of $L^2(\Omega)$. The eigenvalues of a domain are closely related with its geometric properties, constituting an active area of research called *spectral geometry*. A classical example of this relationship is Weyl's law, which relates the asymptotics of the eigenvalues to the volume of the domain, and a later result by McKean and Singer states that the perimeter is also determined by the eigenvalues [20]. More results of this kind can be found in [2] and [24]. Other results about how the geometry of a domain determines its spectrum can be found in Henrot's book [14].

The question of the determination of a domain given the set of its Laplace eigenvalues was posed by Mark Kac in his famous paper "Can one hear the shape of a drum?" [16]. Since then, the answer has been found to be negative in general; in particular, for euclidean polygons, the first example of a pair of non-isometric polygons with the same spectrum is due to Gordon, Webb and Wolpert [13]. However, there are positive results when we restrict the determination to a class of domains, the most successful of which was found by Zelditch [25], who proved spectral determination for analytic domains with two classes of symmetries.

Less is known about domains with more irregular boundaries, the simplest of which are polygons. In the case of triangles, it has been proven that the whole spectrum of the Laplace operator determines the shape of a triangle ([7], with a recent simple proof by [9]), and later Chang and DeTurck proved that only a finite amount of eigenvalues, which depends on $\lambda_1$ and $\lambda_2$, is enough [5]. It is natural to try to improve the result to only a finite and fixed amount of eigenvalues, answering the question "Can a *human* hear the shape of a *triangular* drum?".

Since the space of triangles up to isometries has dimension 3, we would expect that 3 eigenvalues should be enough, and, if so, it is not clear which ones. Antunes and Freitas ([1]) conjectured that indeed the three first eigenvalues $\lambda_1$, $\lambda_2$ and $\lambda_3$ do determine the shape of a triangle. Numerical evidence by themselves seems to indicate that this is not the case for $\lambda_1$, $\lambda_2$ and $\lambda_4$, and in this paper we will prove this fact (Theorem 1.1). This will give an example of an obstruction to determining the shape of a triangle from a finite portion of its spectrum.

The proof of the theorem is computer-assisted: this means that first some topological, analytic and geometric arguments are used to reduce the proof of the theorem to a finite but large number of computations, which are then verified by a computer. The computations are carried in a rigorous way, using the technique of interval arithmetic which keeps track of propagated error bounds for all the computations.

An expanded version of this work, written jointly with Javier Gómez-Serrano and including the codes for the computer verifications, will appear elsewhere and is available as a preprint [11].

# 2. Structure of the proof of Theorem 1.1

By the scaling of the problem, we reduce our search to the set of triangles with a fixed base length (together with additional conditions that ensure that we only consider one triangle for each similarity class); instead of looking for all three eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_4$ to be equal, we just require the quotients $\xi_{21} = \lambda_2/\lambda_1$ and $\xi_{41} = \lambda_4/\lambda_1$ to take the same value. Since the eigenvalues scale by $r^{-2}$ when the lengths of a triangle are scaled by $r$, if two non-congruent triangles are found with the same quotients, there will be a scaling that makes all three eigenvalues coincide.

Fixing the first two vertices of the triangle to be $(0, 0)$ and $(1, 0)$, we use the coordinates $(c_x, c_y)$ of the third vertex to parametrize the search space. Our approach consists in using a topological argument to show that in each of two disjoint regions in this parameter space there is a triangle in which $\xi_{21}$ and $\xi_{41}$ take the same prescribed value. More precisely, we claim that there are two distinct triangles for which $\xi_{21} = \bar{\xi}_{21} := 1.67675$ and $\xi_{41} = \bar{\xi}_{41} := 2.99372$.

Since rigorous calculations with the computer are done using interval arithmetic, we need a topological technique to transform the closed condition into an open condition tolerates error intervals. For that purpose we will use the Poincaré–Miranda theorem (see [19]):

**Theorem 2.1.** *Given two continuous functions $f, g\colon [-1, 1]^2 \to \mathbb{R}$ such that $f(x, y)$ has the same sign as $x$ when $x = \pm 1$ and $g(x, y)$ has the same sign as $y$ when $y = \pm 1$, there exists a point $(x, y) \in [-1, 1]^2$ such that $f(x, y) = g(x, y) = 0$.*

The regions that we will consider are two parallelograms around the points $A = (0.63500, 0.27500)$ and $B = (0.84906, 0.31995)$, designed such that $\xi_{21}$ and $\xi_{41}$ have approximately a constant value each in a pair of opposite edges. Using the computer we will verify that the functions $\xi_{21} - \bar{\xi}_{21}$ and $\xi_{41} - \bar{\xi}_{41}$ each have a constant and opposite sign in opposite edges of the parallelogram, and hence by the theorem, together with the well known domain continuity of eigenvalues, we will conclude that such two distinct triangles exist.

The vectors defining the parallelogram are obtained from the inverse of an approximation of the differential of the $\mathbb{R}^2$-valued funcion $(\xi_{21}, \xi_{41})$ at the points $A$ and $B$, scaled so as to minimize the computation time. This setup is displayed in Figure 1 together with a plot of non rigorous contour lines of the eigenvalue quotients.

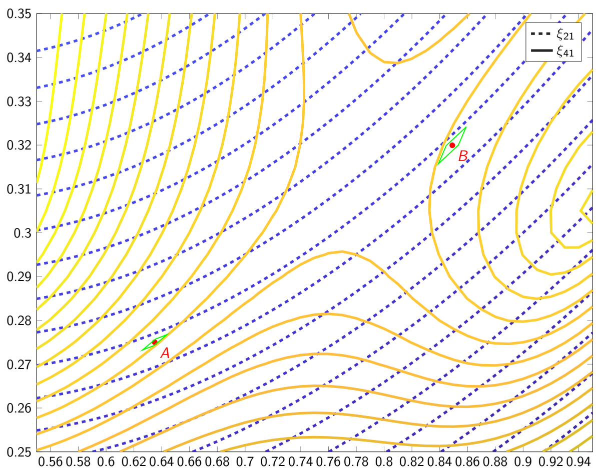


Figure 1: Numerical approximate plot of the quotients $\xi_{21}$ (discontinuous lines) and $\xi_{41}$ (continuous lines) around the region of interest. The validated parallelograms around $A$ and $B$ are shown in green.

The pointwise verification of the values $\xi_{21}$ and $\xi_{41}$ on the edges, which depends on an accurate calculation of $\lambda_i$ for $i = 1, 2, 4$, consists of two steps. The first one, treated in Section 3, is about showing that the computed eigenvalues actually correspond to the ordered ones $\lambda_1$, $\lambda_2$ and $\lambda_4$; in order to do that, we will prove a lower bound for $\lambda_5$ combining techniques from the Finite Element Method with rigorous bounds linking the finite dimensional problem to the infinite dimensional one. The second step consists of finding accurate values of four eigenvalues that lie below the threshold obtained in the first part, which implies that they will indeed have to be the four lowest ones. This is done using the Method of Particular Solutions and recent rigorous bounds based on the $L^2$ norm of the boundary error of candidate approximate eigenfunctions, explained in Section 4.

We emphasize the difficulty of finding the order of an eigenvalue, which is a global problem, compared to the local easier task of refining its value. To the best knowledge of the author, this is the first computer-assisted proof in which these two distinct, local and global methods are used to verify eigenvalues of an operator.

The computer-verified enclosures of $\xi_{21}$ and $\xi_{41}$ explained above are only obtained for a finite set of points. In order to check the hypotheses of the Poincaré–Miranda theorem in all the edges of the parallelograms we will use an argument based on domain monotonicity for the Laplace eigenvalues to propagate the bounds to a neighborhood of the verified points. The details of this part are explained in Section 5. We now explain more about the implementation and execution of the automatic part of the proof.

## 2.1 Implementation of the computer-assisted proof

In the recent years, the application of calculations done by computers to mathematical proofs have become more popular due to the increment of computational resources, but in order to make sure that their results are rigorous, we need to control the errors that floating point arithmetic can accumulate. This is usually done by means of interval arithmetic, in which the data that a computer stores for a real number is an interval (two endpoints, or a midpoint and a radius) of real numbers, stored by two floating point numbers, instead of just one.

Operations between intervals are implemented to return intervals which are guaranteed to contain every possible result when the operands belong to the input intervals. For example, if $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$ are two intervals, their sum will can be given by the interval $[x] + [y] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$ and their product by $[x] \cdot [y] = [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}]$. The same rule applies to function implementations: a function $f$ evaluated on $[x]$ should return an interval containing every $f(x)$ for $x \in [x]$. We refer to the book [23] for an introduction to validated numerics, in which most of the techniques used here are explained, and to [10] for a more specific treatment of computer-assisted proofs in PDE.

The validated computations are performed using the rigorous arithmetic library Arb, developed by Fredrik Johansson [15], which can be found at `http://arblib.org`. Other non-rigorous computations are made using common libraries such as ALGLIB or Boost. The validation of one of the sides of a parallelogram can use approximately from 500 to 2000 points, and the total running time can take from 4 to 14 hours in 120 parallel machines, approximately. These benchmarks are greatly improved in [11] thanks to changes described throughout the text.

# 3. Separation of the first four eigenvalues

In order to find a rigorous lower bound for the fifth eigenvalue of a triangle we will use a recent bound found by Liu [17], which is similar to the one in [6] but simplifies the hypotheses and improves the constant. Both use the non-conforming Finite Element Method of Crouzeix–Raviart; other rigorous bounds with conforming finite elements were explored, like [18], but the bound is worse and the method is harder to implement with validated numerics because its mass matrix is not diagonal.

The Crouzeix–Raviart finite-element method uses a triangulation of the domain $\Omega$, which in our case we will take to be the trivial triangulation given by $N^2$ triangles with sides equal to $1/N$ of the original one and similar to it. The basis functions are indexed by the interior edges of the triangulation: if $E$ is a common edge of triangles $T_1$, $T_2$, the basis function $\psi_E$ is the unique function supported on $T_1 \cup T_2$ such that restricted to each triangle is affine, takes the value 1 in the midpoint of $E$ and the value 0 in the midpoints of the other edges of $T_1$ and $T_2$.

We define the coefficients of the stiffness and mass matrices $\mathbf{A} = (a_{EF})$, $\mathbf{B} = (b_{EF})$ by the bilinear forms

$$a_{EF} = \int_\Omega \nabla\psi_E \cdot \nabla\psi_F, \quad b_{EF} = \int_\Omega \psi_E\psi_F.$$

For our choice of triangulation, $\mathbf{B}$ is simply a multiple of the identity $2|\Omega|\mathbf{I}/(3N^2)$, whereas $\mathbf{A}$ is a sparse matrix. This will allow us to work with a matrix eigenvalue problem for the symmetric matrix $\mathbf{M} = \mathbf{B}^{-1}\mathbf{A}$ instead of a generalized one. The main result that we will use is the following ([17, Thm. 2.1 and Rmk. 2.2]):

**Theorem 3.1.** *Consider a polygonal domain $\Omega$ with a triangulation so that each triangle has diameter at most $h$. Let $\lambda_k$ be the $k$-th eigenvalue of $\Omega$ and $\lambda_{k,h}$ the $k$-th eigenvalue of the Crouzeix–Raviart discretized problem for $\Omega$. Then*

$$\frac{\lambda_{h,k}}{1 + C_h^2\lambda_{h,k}} \leq \lambda_k, \tag{1}$$

*where $C_h \leq 0.1893h$ is a constant.*

In order to be able to deal with approximate eigenvalues we will need in addition the following lemma from [21, Thm. 15.9.1].

**Lemma 3.2.** *Let $(\tilde{\lambda}_h, \tilde{\mathbf{u}}_h)$ be an approximate algebraic eigenpair such that $\tilde{\lambda}_h$ is closer to some $\lambda_h$ than to any other discrete eigenvalue. Suppose that the coefficient vector $\tilde{\mathbf{u}}_h$ is normalised with respect to $\mathbf{B}$, $\|\mathbf{B}\tilde{\mathbf{u}}_h\|_{\mathbf{B}^{-1}} = \|\tilde{\mathbf{u}}_h\|_{\mathbf{B}} = 1$. Then the algebraic residual $\mathbf{r} := \mathbf{A}\tilde{\mathbf{u}}_h - \tilde{\lambda}_h\mathbf{B}\tilde{\mathbf{u}}_h$ satisfies*

$$|\lambda_h - \tilde{\lambda}_h| \leq \|\mathbf{r}\|_{\mathbf{B}^{-1}}.$$

*Remark* 3.3. We can combine Theorem 3.1 with Lemma 3.2 using the monotonicity of (1), using $\lambda_{h,k} - \|\mathbf{r}\|_{\mathbf{B}^{-1}}$ as a lower bound of $\lambda_{h,k}$ instead.

It is easy to obtain estimations $\tilde{\lambda}_h$ with a very small residual; the hardest part here before applying the theorem is to check that they have indeed the correct index, i.e., that they are closer to the appropriate $\lambda_h$ than to any other discrete eigenvalue. Thus we need to control the whole spectrum of the discrete problem.

More precisely, in order to get a lower bound for $\lambda_5$ we need to separate the first 5 eigenvalues from the rest so that we can control them, and in order to do that we will perform Givens rotations until the intervals provided by Gershgorin's theorem can be separated in two disjoint components, one containing the 5 smallest eigenvalues and the other containing the rest. If this holds, then the strong version of

Gershgorin's theorem will guarantee an upper bound for $\lambda_{h,5}$. Therefore, provided that the residuals are all very small and that all approximate eigenvalues are different (which happens in our setting), Lemma 3.2 will guarantee that there are 5 distinct discrete eigenvalues below the upper bound and therefore they will be forced to have the correct indices.

This allows us to verify $\lambda_{h,5}$ with an error only depending on its residual, and using Remark 3.3, get a lower bound of $\lambda_5$. Therefore, the first four eigenvalues can be separated just by checking that they are distinct and smaller than this lower bound.

The Givens rotations must be applied in a rigorous way using interval arithmetic, although the angle of the rotation is chosen in a non rigorous way. The rotations are performed by steps: in each step, several iterations are performed to reduce the upper bound of the lowest 5 Gershgorin intervals below a fixed threshold and to increase the lower bound of the highest Gershgorin intervals above another threshold. These thresholds are improved at each step progressively (the former is reduced, the latter is increased).

The iterations consist in making a Givens rotation to set to zero each off-diagonal entry whose absolute value exceeds the maximum Gershgorin radius allowed (i.e. the difference between the diagonal value and the current threshold) divided by the number of off-diagonal entries. This heuristic tolerates small absolute values in off-diagonal entries and stops when the Gershgorin radius reached is small enough.

The execution of this algorithm for our data required a subdivision into $N^2$ triangles, for $N$ between 18 and 21, and had a running time of between 15 and 45 minutes at a precision of 1024 bits. After the presentation of the thesis, with Gómez-Serrano, we simplified this part and reduced its computation time by showing that we can bound explicitly the difference between the exact diagonal form of the matrix and a nonrigorous diagonalization, provided that it is precise enough. The argument is based on stability bounds of an application of the Gram–Schmidt orthogonalization process to the proposed almost-orthogonal approximate eigenvector basis (see [11] for more details).

# 4. Rigorous eigenvalue bounds for individual triangles

Our approach to find tight bounds for the eigenvalues of triangles uses the Method of Particular Solutions (MPS), introduced by Fox, Henrici and Moler in [8] and more recently revived by Betcke and Trefethen [4]. In this method, a function $u$ is written as a linear combination of functions $\phi_i$ $(1 \leq i \leq N)$ that satisfy pointwise the equation $(\Delta + \lambda)\phi_i = 0$ for a fixed $\lambda$. The coefficients are chosen to optimize the proximity of the function to the eigenspace for the actual eigenvalue $\lambda_j$, in a sense made precise in [4], and this is measured by the least singular value of a certain matrix that involves the values of $u$ at discrete points of the boundary $\partial\Omega$. This parameter is minimized with respect to $\lambda$ by using a golden ratio search. This provides a candidate $\lambda \in \mathbb{R}$ and coefficients $c_i$ for which $u(x) = \sum_{i=1}^{N} c_i \phi_i(x)$ can be computed with arbitrary precision.

The functions that we will use for the MPS consist of two types: the first ones are of the form $\phi(x) = Y_0(\sqrt{\lambda}|x - x_0|)$, for $x_0$ a point outside $\Omega$. The second type of functions are parametrized by a vertex of the triangle and a positive integer $j$, and take the form $\psi_j(r,\theta) = J_{j\alpha}(\sqrt{\lambda}r)\sin(j\alpha\theta)$, where $(r,\theta)$ are the polar coordinates of the point with respect to a vertex in the triangle whose total angle is $\pi/\alpha$, and $\theta$ is measured from an adjacent side. The first kind of functions allow us to approximate the function in the interior of the triangle and near the sides, while the second kind gives the correct asymptotic behavior of the solution near the vertices of the triangle.

We must mention that, shortly before the submission of the thesis, a remarkable paper by Gopal and Trefethen ([12]) introduced a new basis of functions that offers root-exponential convergence, meaning that with a lot fewer functions one could obtain a better fitting in much less time. In the updated version of this work [11], Gómez-Serrano and the author use this so-called *lightning Laplace solver* method to improve the total running time from around a thousand hours to just about 42.

The main tool that we will use to find rigorous bounds for eigenvalues is the $L^2$ bound given by Barnett and Hassell [3]. However, their method is optimized for high eigenvalues, so we will have to adapt some of the steps to our case of small eigenvalues. We summarize the main results that we will use. Let $\Omega$ be a triangle and $u \in C^2(\Omega)$ be nonzero such that $(\Delta + \lambda)u = 0$. Consider the tension

$$t[u] = \frac{\|u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\Omega)}}.$$

Let $\lambda_j$, $u_j$ be the sequence of eigenvalues and eigenfunctions of $\Omega$, satisfying $(\Delta + \lambda_j)u_j = 0$ with Dirichlet null boundary conditions. Let $v_j$ be the normal derivative of $u_j$, defined on $\partial\Omega$. We define the operator

$$A(\lambda) = \sum_{\lambda_j} \frac{v_j\langle v_j, \cdot\rangle}{(\lambda - \lambda_j)^2},$$

and its decomposition as a sum of three:

$$A_{\text{near}}(\lambda) = \sum_{|\lambda - \lambda_j| \leq \sqrt{\lambda}} \frac{v_j\langle v_j, \cdot\rangle}{(\lambda - \lambda_j)^2},$$

$$A_{\text{far}}(\lambda) = \sum_{\lambda/2 \leq \lambda_j \leq 2\lambda,\, |\lambda - \lambda_j| > \sqrt{\lambda}} \frac{v_j\langle v_j, \cdot\rangle}{(\lambda - \lambda_j)^2},$$

$$A_{\text{tail}}(\lambda) = \sum_{\lambda_j < \lambda/2 \text{ or } \lambda_j > 2\lambda} \frac{v_j\langle v_j, \cdot\rangle}{(\lambda - \lambda_j)^2},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $L^2(\partial\Omega)$. This operator is useful because its norm is controlled by the tension (see [3, §3]):

$$t[u]^{-2} \leq \|A(\lambda)\|. \tag{2}$$

Moreover we have the following explicit bounds from [3, Lem. 4.1] and [3, Lem. 4.2]:

$$\|A_{\text{far}}(\lambda)\| \leq C_1, \tag{3}$$

$$\|A_{\text{tail}}(\lambda)\| \leq C_2 \lambda^{-1/2}, \tag{4}$$

with constants $C_1$, $C_2$ given below. For the near term, since we are working with very low eigenvalues, $\sqrt{\lambda}$ is actually small enough that only the summand with $\lambda_j = \lambda$ appears:

$$\|A_{\text{near}}(\lambda)\| = \frac{\|v_j\|^2_{L^2(\partial\Omega)}}{(\lambda - \lambda_j)^2}. \tag{5}$$

For convex domains, like in our case, Section 6 of [3] offers explicit bounds for the constants. By keeping track of all the constants used in their derivation, it is not hard to see that for a triangle with inradius $\rho$, one can take $C_1, C_2 < 28(1+\rho)/\rho$.

Putting (2)–(5) together we have

$$t[u]^{-2} \le \frac{\|v_j\|^2_{L^2(\partial\Omega)}}{(\lambda-\lambda_j)^2} + 7C_\Omega(1+\lambda^{-1/2}). \tag{6}$$

Finally, recall Rellich's formula [22]:

$$\int_{\partial\Omega} (\partial_n u_j)^2 \mathbf{x}\cdot\mathbf{n}\,ds = 2\lambda_j.$$

For our choice of origin of coordinates, this just gives us $\|v_j\|^2_{L^2(\partial\Omega)} = 2\lambda_j/\rho$. Inserting this into (6), we have proved:

**Proposition 4.1.** *The distance $d$ from $\lambda$ to the spectrum of the Laplacian on $\Omega$ can be bounded above by*

$$t[u]^{-2} \le \frac{2\tilde{\lambda}_j}{\rho d^2} + 7C_\Omega(1+\lambda^{-1/2}),$$

*where $\tilde{\lambda}_j$ is an upper bound for $\lambda_j$.*

Thus we must obtain a rigorous upper bound for the $L^2$ norm of the candidate eigenfunction on the boundary of $\Omega$, and a rigorous lower bound for its interior $L^2$ norm, in order to use the previous proposition and deduce the existence of an actual eigenvalue near the candidate one.

## 4.1 Upper bound of the boundary norm

This computation is done by dividing the sides of the triangle into many small intervals, in positions given by Chebyshev nodes, and in each of them performing a validated computation using Taylor series: the Taylor polynomial of the function at the center point is evaluated in the whole interval, and to this value the remainder of Taylor's theorem is added. A validated enclosure for this remainder consists of the Taylor polynomial of the function at the whole interval evaluated in the whole interval.

When this computation exceeds a threshold in absolute value (in our case, $10^{-5}$), the interval is split in half and the validating function is called recursively for the two halves. In the end, all contributions from all intervals are added up to get the $L^2$ bound. This calculation takes approximately 10 minutes per point at a precision of 128 bits, using a total of 317 charge basis functions and 15 vertex basis functions.

## 4.2 Lower bound of the interior norm

This bound is obtained by using a grid of $8\times 8$ small triangles that occupy a smaller triangle of side 0.8 times the original one (hence the area of each triangle is $0.01|\Omega|$). Figure 2 displays the grid for the plot of the first and the fourth eigenfunction of triangle $B$. In each of the triangles a lower bound of the absolute value of $u$ is obtained using the same method as above (Taylor series bounds and splitting recursively).

Whenever we obtain a validated estimate, say $u \geq a > 0$, on $\partial T$, where $T$ is one of the small triangles in the grid, we can extend this inequality to the whole $T$ by using the minimum principle. More precisely, it is enough to show that $-\Delta u \geq 0$ in $T$ to get that the minimum of $u$ is in $\partial T$ and hence is at least $a$. If this does not hold, it means that $\lambda u = -\Delta u < 0$ at some point inside $T$. This means that the open set $U = \{u < 0\} \cap T \subset T$ has $\lambda$ as a Dirichlet eigenvalue, and hence by the Faber–Krahn inequality, we get a lower bound for its area:

$$0.01|\Omega| = |T| \geq |U| \geq \frac{\pi j_{0,1}^2}{\lambda} > \frac{18.1684}{\lambda}.$$

This is a contradiction by orders of magnitude for our triangles ($|\Omega| \leq 0.25$ and $\lambda < 1000$).

The calculation of this part also takes approximately 10 minutes per point at a precision of 128 bits, using a total of 317 charge basis functions and 15 vertex basis functions.

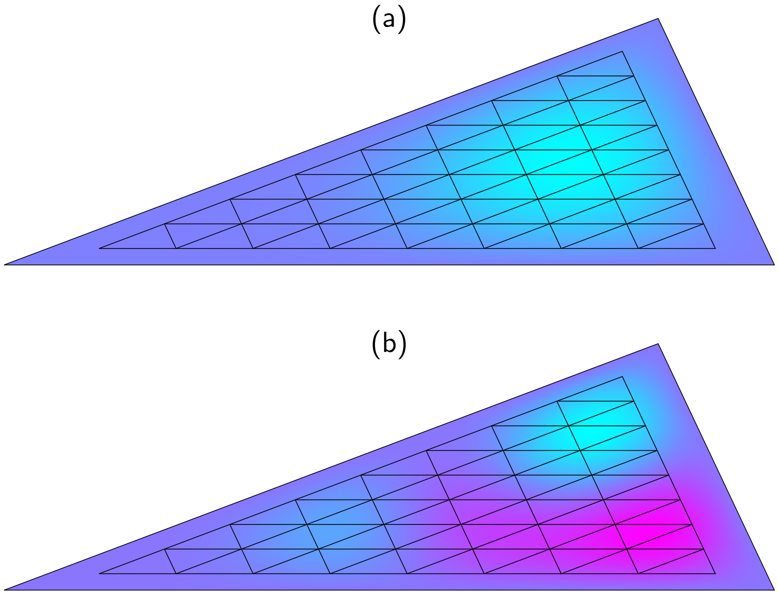


Figure 2: Grid used to validate a lower bound for $\|u\|_{L^2(\Omega)}$ for triangle $B$, with (a) the first eigenfunction and (b) the second eigenfunction plotted on top.

# 5. Extension of the bounds to a region of triangles

Our goal is to propagate the rigorous bounds of an eigenvalue of the Laplacian of a triangle with Dirichlet boundary conditions to a neighborhood of triangles. In the original version of the thesis, this was done using a continuity argument based on the fact that an operator norm bound of the difference of two compact operators (in this case, the inverse of the Laplacian and of a deformed version of it associated to a neighboring triangle) translates into a bound of the difference of all their respective eigenvalues. We later realized that obtaining an explicit bound for the difference of such operators was harder than we thought, and discovered another method which is conceptually much simpler and surprisingly propagates the bound into longer intervals. Therefore we will just sketch this simpler method, and refer to [11] for more details.

**Lemma 5.1.** *Let $T$ and $T'$ be two triangles, whose vertices are $A = (0,0)$, $B = (1,0)$, and $C = (c_x, c_y)$, $C' = (c'_x, c'_y)$ respectively ($c_y, c'_y > 0$). Consider the cross products $p = \overrightarrow{AC} \times \overrightarrow{AC'} = c_x c'_y - c_y c'_x$ and $q = \overrightarrow{BC} \times \overrightarrow{BC'} = (c_x - 1)c'_y - c_y(c'_x - 1)$. Then,*

*(i) if both $p, q < 0$, there is a homothety of $T'$ by a factor $1 - p/c'_y$ that contains $T$;*

*(ii) if both $p, q > 0$, there is a homothety of $T'$ by a factor $1 + q/c'_y$ that contains $T$.*

*Proof.* The proof is very similar in the two cases, so we will only do it for the first one. We want to find the homothety of scale $1 + r$ that keeps the vertex $B$ of triangle $T'$ fixed and such that the image of its opposite side contains vertex $C$ of $T$. The condition becomes simpler once we apply an inverse homothety to $T$ and $T'$, so that it results in the points $A$, $C''' = (C + rB)/(1 + r)$, $C'$ being aligned. The solution is $r = -p/c'_y$, which is positive by our condition. Moreover, triangle $T$ lies below this homothety of $T'$ because the vectors $\overrightarrow{BC}$ and $\overrightarrow{BC'}$ are in the correct orientation due to the condition $q < 0$. This suffices to check that $T$ is contained in this homothety. □

**Lemma 5.2.** *With the same notation as in Lemma 5.1,*

*(i) if $p > 0$ and $q < 0$, then $T \subset T'$;*

*(ii) if $p < 0$ and $q > 0$, there is a homothety of $T'$ by a factor $c_y/c'_y > 1$ that contains $T$.*

*Proof.* In the first case, the conditions on the signs of the cross products of the side vectors is equivalent to $T$ being contained in $T'$. In the second case, the relative orientations of the sides guarantee that a homothetic triangle to $T'$ of the same height as $T$ whose top vertex coincides with $C$ will contain $T$, and the ratio of this homothety is clearly $c_y/c'_y$. □

Using the reversed inclusions from the previous lemmas, an easy but tedious calculation which distinguishes the two cases above leads to the following result, that can be applied directly to propagate a bound on $\xi_{21}$ or $\xi_{41}$ to a neighborhood of a triangle.

**Lemma 5.3.** *Let $T$ be a triangle as above, and consider perturbations of the third vertex of the form $C + tv$ defining triangles $T^{(t)}$, for $t \in [-\ell, \ell]$, where $v = (v_x, v_y)$. Let $\lambda_n$, $\lambda_n^{(t)}$ be the n-th Dirichlet eigenvalues of triangles $T$, $T^{(t)}$, respectively, and define $\xi_{n1}^{(t)}$ as the obvious eigenvalue quotient. Then we distinguish two cases depending on $p_v = \overrightarrow{AC} \times v$ and $q_v = \overrightarrow{BC} \times v$:*

*(i) if $p_v$ and $q_v$ both have the same sign, then for all $t \in [-\ell, \ell]$*

$$|\xi_{n1}^{(t)} - \xi_{n1}| \leq \xi_{n1} \left[ \left(1 + \ell \frac{|p_v|}{c_y - \ell|v_y|}\right)^2 \left(1 + \ell \frac{|q_v|}{c_y - \ell|v_y|}\right)^2 - 1 \right];$$

*(ii) if $p_v$ and $q_v$ have different signs, then for all $t \in [-\ell, \ell]$*

$$|\xi_{n1}^{(t)} - \xi_{n1}| \leq \xi_{n1} \left[ \left( \frac{c_y}{c_y - \ell|v_y|} \right)^2 - 1 \right].$$

## Expression of gratitude

First of all, I would like to thank my supervisor Javier Gómez-Serrano for hosting me at Princeton University, suggesting me the problem, providing me references, giving me a lot of guidance and sharing with me many useful discussions. I am very grateful to the Mathematics Department of Princeton University, in particular the Graduate Program, for funding my university fee and allowing my stay as a visitor researcher. I would also like to thank CFIS and the administrators of the Mobility program for giving me the opportunity to do this research in another university and funding me. I also thank the MOBINT scholarship for providing partial financial support for my stay.

## References

[1] P.R.S. Antunes, P. Freitas. On the inverse spectral problem for Euclidean triangles, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **467**(2130) (2011), 1546–1562.

[2] T.P. Branson, P.B. Gilkey. The asymptotics of the Laplacian on a manifold with boundary, *Comm. Partial Differential Equations* **15**(2) (1990), 245–272.

[3] A.H. Barnett, A. Hassell. Boundary quasi-orthogonality and sharp inclusion bounds for large Dirichlet eigenvalues, *SIAM J. Numer. Anal.* **49**(3) (2011), 1046–1063.

[4] T. Betcke, L.N. Trefethen. Reviving the method of particular solutions, *SIAM Rev.* **47**(3) (2005), 469–491.

[5] P.-K. Chang, D. DeTurck. On hearing the shape of a triangle, *Proc. Amer. Math. Soc.* **105**(4) (1989), 1033–1038.

[6] C. Carstensen, J. Gedicke. Guaranteed lower bounds for eigenvalues, *Math. Comp.* **83**(290) (2014), 2605–2629.

[7] C. Durso. On the inverse spectral problem for polygonal domains, Ph.D. thesis, Massachusetts Institute of Technology, 1988.

[8] L. Fox, P. Henrici, C. Moler. Approximations and bounds for eigenvalues of elliptic operators, *SIAM J. Numer. Anal.* **4**(1) (1967), 89–102.

[9] D. Grieser, S. Maronna. Hearing the shape of a triangle, *Notices Amer. Math. Soc.* **60**(11) (2013), 1440–1447.

[10] J. Gómez-Serrano. Computer-assisted proofs in PDE: a survey, *SeMA J.* **76**(3) (2019), 459–484.

[11] J. Gómez-Serrano, G. Orriols. Any three eigenvalues do not determine a triangle, to appear in *J. Differential Equations*. Available at arXiv:1911.06758.

[12] A. Gopal, L.N. Trefethen. Solving Laplace problems with corner singularities via rational functions, *SIAM J. Numer. Anal.* **57**(5) (2019), 2074–2094.

[13] C. Gordon, D.L. Webb, S. Wolpert. One cannot hear the shape of a drum, *Bull. Amer. Math. Soc. (N.S.)* **27**(1) (1992), 134–138.

[14] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.

[15] F. Johansson. Arb: efficient arbitrary-precision midpoint-radius interval arithmetic, *IEEE Trans. Comput.* **66**(8) (2017), 1281–1292.

[16] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly* **73**(4), part II (1966), 1–23.

[17] X. Liu. A framework of verified eigenvalue bounds for self-adjoint differential operators, *Appl. Math. Comput.* **267** (2015), 341–355.

[18] X. Liu, S. Oishi. Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape, *SIAM J. Numer. Anal.* **51**(3) (2013), 1634–1654.

[19] C. Miranda. Un'osservazione su un teorema di Brouwer, *Boll. Un. Mat. Ital. (2)* **3** (1940), 5–7.

[20] H.P. McKean, Jr., I.M. Singer. Curvature and the eigenvalues of the Laplacian, *J. Differential Geometry* **1**(1) (1967), 43–69.

[21] B.N. Parlett. *The symmetric eigenvalue problem*, Prentice-Hall Series in Computational Mathematics, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980.

[22] F. Rellich. Darstellung der Eigenwerte von$\Delta$ $u+\lambda u = 0$ durch ein Randintegral, *Math. Z.* **46** (1940), 635–636.

[23] W. Tucker. *Validated numerics. A short introduction to rigorous computations*, Princeton University Press, Princeton, NJ, 2011.

[24] M. van den Berg, S. Srisatkunarajah. Heat equation for a region in $\mathbf{R}^2$ with a polygonal boundary, *J. London Math. Soc. (2)* **37**(1) (1988), 119–127.

[25] S. Zelditch. Inverse spectral problem for analytic domains. II. $\mathbb{Z}_2$-symmetric domains, *Ann. of Math. (2)* **170**(1) (2009), 205–269.